Whitham equations in the AKNS scheme

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Abstract

A general method is formulated for the derivation of the modulation Whitham equations of periodic solutions to integrable equations described by the AKNS scheme. The method provides the Whitham equations in a form convenient for applications. It is illustrated by the derivation of a new form of the Whitham equations corresponding to the modulation of one- and two-phase periodic solutions of the nonlinear Schrödinger equation.

1. Introduction

The Whitham averaging method was applied first [1] to the periodic solution of the Korteweg-de Vries (KdV) equation. Later it was found [2] that this method could be rather naturally formulated in terms of the inverse scattering transform method of integration of nonlinear evolution equations. Such a formulation permitted one to obtain the Whitham equations for multi-phase periodic solutions of the KdV equation [2], sine- and sinh-Gordon equations [3], and nonlinear Schrödinger (NLS) equation [4].

In Refs. [5–8] a simple method of derivation of the Whitham equations has been suggested and applied to a number of physical systems. Now we want to demonstrate its applicability to a wide class of integrable equations described by the AKNS scheme [9]. This method yields the Whitham equations in a form convenient for their integration with the use of the generalized hodograph transform [10–12].

2. General theory

Let the evolution equations be represented as a compatibility condition of two systems of linear equations,

\[
\begin{align*}
\frac{\partial \psi_1}{\partial x} &= F \psi_1 + G \psi_2, \\
\frac{\partial \psi_2}{\partial x} &= H \psi_1 - F \psi_2,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial \psi_1}{\partial t} &= A \psi_1 + B \psi_2, \\
\frac{\partial \psi_2}{\partial t} &= C \psi_1 - A \psi_2,
\end{align*}
\]

where the coefficients depend on the arbitrary spectral parameter \(\lambda\) and on functions \(u_k(x, t)\) of which the evolution is governed by the equations under consideration. In terms of the above coefficients these evolution equations can be written as

\[
\begin{align*}
\frac{\partial F}{\partial t} - \frac{\partial A}{\partial x} + CG - BH &= 0, \\
\frac{\partial G}{\partial t} - \frac{\partial B}{\partial x} + 2BF - 2AG &= 0, \\
\frac{\partial H}{\partial t} - \frac{\partial C}{\partial x} + 2AH - 2CF &= 0.
\end{align*}
\]
Systems (2.1) and (2.2) have two basic solutions \((\psi_1, \psi_2)\) and \((\varphi_1, \varphi_2)\), and we can construct from these two "spinors" the "vector" with spherical components

\[
\begin{align*}
\psi &= -\frac{1}{2}(\psi_1 \varphi_2 + \psi_2 \varphi_1), \quad \varphi = \psi_1 \varphi_1, \\
\chi &= -\psi_2 \varphi_2.
\end{align*}
\]

They satisfy the following linear equations,

\[
\begin{align*}
\frac{\partial \psi}{\partial x} &= -iH \varphi + iG \chi, \quad \frac{\partial \varphi}{\partial t} = -iC \psi + iB \chi, \\
\frac{\partial \varphi}{\partial x} &= 2iG \psi + 2F \varphi, \quad \frac{\partial \psi}{\partial t} = 2iB \varphi + 2A \psi, \\
\frac{\partial \chi}{\partial x} &= -2iH \varphi - 2F \chi, \quad \frac{\partial \varphi}{\partial t} = -2iC \psi - 2A \varphi.
\end{align*}
\]

(2.4)

It is easy to check that the length of the vector (2.4),

\[f^2 - g \chi = P(\lambda),\]

(2.6)
does not depend on \(x\) and \(t\). Periodic solutions are distinguished by the condition that \(P(\lambda)\) be a polynomial in \(\lambda\). The function \(g\) can be looked for in the form

\[g = \phi \prod_{j=1}^{m} [\lambda - \mu_j(x, t)],\]

(2.7)

where \(m\) is a genus of the hyperelliptic curve

\[w^2 = P(\lambda),\]

(2.8)

\(\mu_j\) are the points of the so-called auxiliary spectrum, and the function \(\phi\) is such that \(G/\phi\) and \(B/\phi\) can be expressed in terms of \(\mu_j(x, t)\) by means of identities (conservation laws) following from (2.6) after comparing the coefficients of \(\lambda^k\) on both sides of (2.6),

\[G/\phi = \bar{G}(\mu_1, ..., \mu_m), \quad B/\phi = \bar{B}(\mu_1, ..., \mu_m).\]

(2.9)

Substitution of (2.7) into (2.5) at \(\lambda = \mu_j, j = 1, .., m\), yields the equations for \(\mu_j(x, t)\) in the Dubrovin form

\[
\begin{align*}
\frac{\partial \mu_j}{\partial x} &= \frac{2i \bar{G}\sqrt{P(\mu_j)}}{\prod_{k \neq j} (\mu_j - \mu_k)}, \\
\frac{\partial \mu_j}{\partial t} &= \frac{2i \bar{B}\sqrt{P(\mu_j)}}{\prod_{k \neq j} (\mu_j - \mu_k)}.
\end{align*}
\]

(2.10)

Eqs. (2.10) can be integrated with the help of the Abel transform which leads to the expressions for \(\mu_j\) in terms of theta functions depending on phase variables

\[\theta_j = \kappa_j x + \omega_j t + \theta_0, \quad j = 1, .., m,\]

(2.11)

where \(\kappa_j\) and \(\omega_j\) are determined by integrals over certain cycles on the Riemann surface of the curve (2.8) (see, e.g., Refs. [2,3,14-16]).

The method of derivation of the Whitham equations suggested in Refs. [5-8] is based on the following form of generating functions of conservation laws in the AKNS scheme,

\[
\begin{align*}
\frac{\partial G}{\partial t} \varphi + \frac{\partial B}{\partial x} \varphi &= 0, \\
\frac{\partial H}{\partial t} \chi - \frac{\partial C}{\partial x} \chi &= 0.
\end{align*}
\]

(2.12)

(2.13)

Their validity can be easily proved with the use of (2.3) and (2.5).

Averaging of (2.12), (2.13) immediately leads to the Whitham equations in an invariant Riemann form. Let the polynomial \(P(\lambda)\) have the zeros \(\lambda_i, i = 1, ..., n\), which play the role of the Riemann invariants. Before averaging the vector (2.4) should be normalized according to the condition of unit length \(f^2 - g \chi = 1\) (see Refs. [2,3]). Then Eq (2.12) can be written as

\[
\frac{\partial}{\partial t} \frac{G}{\prod_{j=1}^{m} (\lambda - \mu_j)} - \frac{\partial}{\partial x} \frac{\bar{B}\sqrt{P(\lambda)}}{\prod_{j=1}^{m} (\lambda - \mu_j)} = 0.
\]

(2.14)

In the case of a slowly nonuniform solution the parameters \(\lambda_i, i = 1, ..., n\), become functions of slow variables \(X, T\), and they change little over each wavelength and each period of a multi-phase quasi-periodic solution. This permits one to average (2.14) over the fast space variable \(x\) according to the rule [2,3]

\[
\langle \Phi \rangle = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \Phi \, dx
\]

(2.15)

The averaging over the space variable can be replaced by the averaging over phase variables (2.11) provided the spatial wave numbers \(\kappa_i\) are incommensurate [2,3],

\[
\langle \Phi \rangle = \frac{1}{(2\pi)^m} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \Phi \, d\theta_1 \cdots d\theta_m.
\]

(2.16)
where integration over $\theta$-variables can be changed to integration over $\mu$-variables,

$$
\langle \Phi \rangle = \frac{1}{(2\pi)^m} \int_{C_1} \cdots \int_{C_m} \Phi \frac{\partial}{\partial \mu} \, d\mu_1 \cdots d\mu_m,
$$

(2.17)

where the Jacobian $\partial (\theta) / \partial (\mu)$ should be found with the help of (2.10), (2.11); $C_j, j = 1, \ldots, m,$ are the cycles defining the solution of Eqs. (2.10) according to the Abel transform. The condition for vanishing of the coefficients of the singular terms resulting from differentiation of $\sqrt{P(\lambda)}$ in (2.14) with respect to $X$ and $T$ yields the equations

$$
\frac{\partial \lambda_i}{\partial T} + v_i \frac{\partial \lambda_i}{\partial X} = 0, \quad i = 1, \ldots, n.
$$

(2.18)

Thus, we obtain the desired Whitham equations for the Riemann invariants $\lambda_i$,

$$
\frac{\partial \lambda_i}{\partial T} + v_i \frac{\partial \lambda_i}{\partial X} = 0, \quad i = 1, \ldots, n,
$$

(2.19)

where the characteristic speeds are equal to

$$
v_i = -\frac{I_2(\lambda_i)}{I_1(\lambda_i)},
$$

(2.20)

$$
I_1(\lambda_i) = \int_{C_1} \cdots \int_{C_m} \frac{\mathcal{G}(\mu_1, \ldots, \mu_m)}{\prod_{j=1}^m (\lambda_i - \mu_j)} \frac{\partial}{\partial \mu} \, d\mu_1 \cdots d\mu_m,
$$

(2.21)

$$
I_2(\lambda_i) = \int_{C_1} \cdots \int_{C_m} \frac{\mathcal{B}(\mu_1, \ldots, \mu_m)}{\prod_{j=1}^m (\lambda_i - \mu_j)} \frac{\partial}{\partial \mu} \, d\mu_1 \cdots d\mu_m.
$$

(2.22)

In all known examples these integrals can be easily evaluated by reducing the multiple integrals to the single ones.

3. Example

As a simple concrete example let us consider the derivation of the Whitham equations for one- and two-phase periodic solutions of the NLS equation,

$$
iu_t + uu_x + 2|u|^2u = 0.
$$

(3.1)

In this case we have (see, e.g., Refs. [15,16])

$$
G = iu, \quad B = -u_x + 2\mu \lambda u,
$$

(3.2)

It is easy to find from (2.5) and (2.7) that

$$
u_x = 2iu \left( \sum_{j=1}^m \mu_j - s_1/2 \right), \quad s_1 = \sum_{i=1}^n \lambda_i,
$$

(3.3)

and hence

$$
\mathcal{G} = 1, \quad \mathcal{B} = s_1 + 2 \left( \lambda - \sum_{j=1}^m \mu_j \right).
$$

(3.4)

The necessary Jacobian was calculated in Refs. [2,3] and equals

$$
\frac{1}{(2\pi)^m} \frac{\partial (\theta)}{\partial (\mu)} = \frac{1}{V} \prod_{i>j}^k (\mu_i - \mu_k),
$$

(3.5)

where $V$ is a constant factor which cancels after substitution into (2.19).

In the one-phase case ($m = 1$) the polynomial $P(\lambda)$ is of fourth degree and we have four Riemann invariants $\lambda_i$,

$$
P(\lambda) = \prod_{i=1}^4 (\lambda - \lambda_i).
$$

(3.6)

Integrals (2.21) and (2.22) are equal to

$$
I_1(\lambda_i) = \int_C \frac{d\mu}{(\lambda_i - \mu) \sqrt{P(\mu)}},
$$

(3.7)

$$
I_2(\lambda_i) = s_1 I_1(\lambda_i) + 2L,
$$

where

$$
L = \int_C \frac{d\mu}{\sqrt{P(\mu)}}
$$

(3.8)

is the wavelength (multiplied by a constant factor inessential in this case) of the periodic solution. Using the obvious formula

$$
I_1(\lambda_i) = -2 \frac{\partial L}{\partial \lambda_i},
$$

(3.9)

we find that $\lambda_i$ obey the Whitham equations (2.18) with characteristic speeds.
\[ v_i = -\left[ s_1 - \left( \frac{\partial \ln L}{\partial \lambda_j} \right)^{-1} \right] . \] (3.10)

If we express the wavelength (3.8) in terms of a complete elliptic integral of the first kind and substitute it into (3.10), then we will reproduce the well known formulae obtained in Ref. [4].

For the first time a formula of the type (3.10) was obtained in Ref. [6] in the case of the derivative nonlinear Schrödinger equation and in Refs. [7,8] in the case of the Heisenberg continuous classical spin model. Later such a formula was derived by a different method in Refs. [17,18] in the case of the KdV equation. This form of presentation of velocities \( v_i \) has been found to be rather useful for integration of the Whitham equations by the generalized hodograph transform (see, e.g., Refs. [17,19,20]).

In the two-phase case we have six Riemann invariants \( \lambda_i \),

\[ P(\lambda) = \prod_{i=1}^{6} (\lambda - \lambda_i) , \] (3.11)

so that

\[ I_1(\lambda_i) = \int_{C_1} \int_{C_2} \frac{(\mu_2 - \mu_1)}{(\lambda_2 - \mu_1)(\lambda_1 - \mu_2)[P(\mu_1)P(\mu_2)]^{1/2}} \frac{d\mu_1}{C_1} \frac{d\mu_2}{C_2} \]

\[ = \int_{C_1} \frac{d\mu}{\sqrt{P(\mu)}} \int_{C_2} \frac{d\mu}{(\lambda_1 - \mu)\sqrt{P(\mu)}} \]

\[ - \int_{C_2} \frac{d\mu}{\sqrt{P(\mu)}} \int_{C_1} \frac{d\mu}{(\lambda_1 - \mu)\sqrt{P(\mu)}} \] (3.12)

\[ I_2(\lambda_i) = s_1 I_1(\lambda_i) \]

\[ + 2 \int_{C_1} \int_{C_2} \frac{(\lambda_1 - \mu_1 - \mu_2)(\mu_2 - \mu_1)}{(\lambda_1 - \mu_1)(\lambda_1 - \mu_2)[P(\mu_1)P(\mu_2)]^{1/2}} \frac{d\mu_1}{C_1} \frac{d\mu_2}{C_2} \]

\[ = s_1 I_1(\lambda_i) - 2 \int_{C_1} \frac{d\mu}{\sqrt{P(\mu)}} \int_{C_2} \frac{d\mu}{(\lambda_1 - \mu)\sqrt{P(\mu)}} \]

\[ + 2 \int_{C_2} \frac{d\mu}{\sqrt{P(\mu)}} \int_{C_1} \frac{d\mu}{(\lambda_1 - \mu)\sqrt{P(\mu)}} \] (3.13)

Let us introduce the hyperelliptic integrals

\[ U_{ij} = \int_{C_1} \frac{\mu^{j-1}}{\sqrt{P(\mu)}} d\mu \] for \( i = 1, 2 \).

Then (3.12) and (3.13) can be easily expressed in terms of these integrals and we arrive at the formulae

\[ v_i = -\left[ s_1 - \frac{U_{12}(\partial U_{21}/\partial \lambda_i) - U_{22}(\partial U_{11}/\partial \lambda_i)}{U_{11}(\partial U_{21}/\partial \lambda_i) - U_{21}(\partial U_{11}/\partial \lambda_i)} \right] , \]

\( i = 1, \ldots, 6 \). (3.15)

for characteristic speeds, which seem to be simple enough for applications.

4. Conclusion

Examples of the derivation of the Whitham equations in Refs. [5-8] and in this article show that the method under consideration is rather effective and it provides the desired equations in a form suitable for applications.

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References

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